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Ablowitz–Ladik hierarchy and two-component Toda lattice hierarchy

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Abstract

It is shown in the language of free fermions and the τ functions that the Ablowitz–Ladik (AL) hierarchy and the relativistic Toda lattice hierarchy arise from the $A_1^{(1)}$ reduction of the two-component Toda lattice hierarchy. This result gives us a simple explanation of the reason why the AL hierarchy includes important integrable systems such as the discrete nonlinear Schrödinger equation, the relativistic Toda lattice equation, the Toda field equation, the Davey–Stewartson equation, etc. We also propose a new Bäcklund transformation (or a discretized time system) for the AL hierarchy.

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1. Introduction

In the 1980s, Sato and his colleagues investigated the Kadomtsev–Petviashvili (KP) equation and discovered the KP hierarchy. They showed that the τ functions of the KP hierarchy are parametrized by the universal Grassmann manifold. Date *et al* explained the KP hierarchy using the theory of free fermions [1]. They also demonstrated that the nonlinear Schrödinger equation is contained in the $A_1^{(1)}$ reduction of the two-component KP hierarchy. Ueno and Takasaki constructed the Toda lattice hierarchy [2] and showed that it has a similar structure to the KP hierarchy. Takebe represented the Toda lattice hierarchy in the language of free fermions [3, 4].

The Ablowitz–Ladik (AL) hierarchy is an infinite set of differential-difference equations which includes the spatially discrete nonlinear Schrödinger (sdNLS) equation [5]. This system has been studied by many authors as a typical spatial discretization of integrable systems [6]. Vekslerchik [7–9] showed that the AL hierarchy contains many important integrable equations such as the Toda field equation, the KP equation, the spatially discrete modified

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Korteweg–de Vries (sdmKdV) equation, the sdNLS equation, the Davey–Stewartson equation, etc. He called it the 'universality' [8, 9] of the AL hierarchy, and derived a functional representation of this system [10, 11].

On the other hand, the relativistic Toda lattice equation is proposed by Ruijsenaars [12] as a particular generalization of the Toda lattice equation, and it is often called the Ruijsenaars–Toda (RT) equation. Kharchev *et al* discussed the relation between the RT hierarchy and the AL hierarchy [13]. They showed that the AL hierarchy is derived from the one-component Toda lattice hierarchy with a reduction.

In this paper, we shall show in terms of free fermion operators that the AL hierarchy and the RT hierarchy arise from the $A_1^{(1)}$ reduction of the two-component Toda lattice (2cTL) hierarchy. This makes us understand the τ function of the AL hierarchy (and the RT hierarchy) by means of the affine Lie algebra that acts on the universal Grassmann manifold. At the same time, the origin of the 'universality' of the AL hierarchy will be clarified.

This paper is planned as follows. In section 2, we formulate the two-component Toda lattice hierarchy and reduce its symmetry to the affine Lie algebra $A_1^{(1)}$. The bilinear integral identity is also presented. In section 3, we introduce three τ functions. The relations among these τ functions are derived from the bilinear integral identity, and they are shown to be equivalent to the scattering problem of the AL hierarchy. The functional representation by Vekslerchik is also recovered. Section 4 provides the τ function of the soliton solution of the RT hierarchy. Section 5 is for conclusion and discussion.

2. Two-component Toda lattice hierarchy and its reduction

The 2cTL hierarchy arises as a multicomponent generalization of the one-component Toda lattice hierarchy. Two formalizations are known for multicomponent generalization of the KP hierarchy. One is by Sato [14]; the other is by Date *et al* [15]. The multicomponent generalization of the TL hierarchy by Ueno and Takasaki [2] is based on the former. In this paper, we generalize the TL hierarchy parallel to the latter. Indeed, we formulate the 2cTL hierarchy and its reduction to $A_1^{(1)}$ by means of the theory of free fermions.

Let $\psi_i^{(\alpha)}$ and $\psi_i^{*(\alpha)}$ ($\alpha = 1, 2, i \in \mathbb{Z} + 1/2$) stand for free fermion operators with the anticommutation rules

$$\left[\psi_i^{(\alpha)}, \psi_j^{(\beta)}\right]_+ = \left[\psi_i^{*(\alpha)}, \psi_j^{*(\beta)}\right]_+ = 0 \qquad \left[\psi_i^{(\alpha)}, \psi_j^{*(\beta)}\right]_+ = \delta_{\alpha,\beta}\delta_{i+j,0}$$

where $\delta_{i,j}$ is the Kronecker delta:

$$\delta_{i,j} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases} \tag{1}$$

Consider an infinite number of 'times' $x^{(\alpha)} = \left(x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots\right)$ and $\bar{x}^{(\alpha)} = \left(\bar{x}_1^{(\alpha)}, \bar{x}_2^{(\alpha)}, \ldots\right)$ ($\alpha = 1, 2$), and the Hamiltonians associated with these times,

$$H_{\pm m}^{(\alpha)} = \sum_{i\in\mathbb{Z}+1/2} : \psi_{-i}^{(\alpha)}\psi_{i\pm m}^{*(\alpha)} : \qquad (\alpha = 1, 2; m \in \mathbb{N}). \tag{2}$$

Here,

$$\mathcal{H}(x^{(1)}, x^{(2)}) = \sum_{\alpha=1,2}\sum_{m=1}^{\infty} x_m^{(\alpha)} H_m^{(\alpha)} \qquad \bar{\mathcal{H}}(\bar{x}^{(1)}, \bar{x}^{(2)}) = \sum_{\alpha=1,2}\sum_{m=1}^{\infty} \bar{x}_m^{(\alpha)} H_{-m}^{(\alpha)}.$$

We define the counterpart of the generalized vacuum vectors $\langle l_1, l_2|$ and $|l_2, l_1\rangle$ ($l_1, l_2 \in \mathbb{Z}$) by

$$\langle l_1, l_2| = \langle\text{vac}|\Upsilon_{l_1}^{*(1)}\Upsilon_{l_2}^{*(2)} \qquad |l_2, l_1\rangle = \Upsilon_{l_2}^{(2)}\Upsilon_{l_1}^{(1)}|\text{vac}\rangle \tag{3}$$

with

$$\Upsilon_l^{(\alpha)} = \begin{cases} \psi^{(\alpha)}_{-l+1/2}\dots\psi^{(\alpha)}_{-1/2} & (l>0)\\ \mathbf{1} & (l=0)\\ \psi^{*(\alpha)}_{l+1/2}\dots\psi^{*(\alpha)}_{-1/2} & (l<0)\end{cases} \qquad \Upsilon_l^{*(\alpha)} = \begin{cases} \psi^{*(\alpha)}_{1/2}\dots\psi^{*(\alpha)}_{l-1/2} & (l>0)\\ \mathbf{1} & (l=0)\\ \psi^{(\alpha)}_{1/2}\dots\psi^{(\alpha)}_{-l-1/2} & (l<0)\end{cases}$$

and $\langle \mathrm{vac}|\mathrm{vac}\rangle = 1$.

Quadratic forms of free fermions,

$$\sum_{\alpha,\beta=1,2}\sum_{i,j\in\mathbb{Z}+1/2} :\psi^{(\alpha)}_{-i}\psi^{*(\beta)}_{j}: a^{(\alpha,\beta)}_{i,j} \qquad \left(a^{(\alpha,\beta)}_{i,j}=0 \qquad \text{for} \quad |i-j|<\exists N\in\mathbb{N}\right)$$

constitute the affine Lie algebra $gl(\infty)$; for any element of the corresponding group, $g\in GL(\infty)$,

$$\sum_{\alpha=1,2}\sum_{i\in\mathbb{Z}+1/2}\psi^{(\alpha)}_{-i}g\otimes\psi^{*(\alpha)}_{i}g = \sum_{\alpha=1,2}\sum_{i\in\mathbb{Z}+1/2} g\psi^{(\alpha)}_{-i}\otimes g\psi^{*(\alpha)}_{i}. \tag{4}$$

Introducing fermion fields $\psi^{(\alpha)}(k)$ and $\psi^{(\alpha)*}(k)$ with $k\in\mathbb{C}$ by

$$\psi^{(\alpha)}(k) = \sum_{i\in\mathbb{Z}+1/2}\psi^{(\alpha)}_{i}k^{-i-1/2} \qquad \psi^{*(\alpha)}(k) = \sum_{i\in\mathbb{Z}+1/2}\psi^{*(\alpha)}_{i}k^{-i-1/2}$$

we obtain the bilinear integral identity

$$\sum_{\alpha=1,2}\oint\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\langle u|\psi^{(\alpha)}(\lambda)g|v\rangle\langle u'|\psi^{*(\alpha)}(\lambda)g|v'\rangle = \sum_{\alpha=1,2}\oint\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\langle u|g\psi^{(\alpha)}(\lambda)|v\rangle\langle u'|g\psi^{*(\alpha)}(\lambda)|v'\rangle \tag{5}$$

for any bras $\langle u|$ and $\langle u'|$ and any kets $|v\rangle$ and $|v'\rangle$. Note that, unlike the case of the KP hierarchy, the rhs does not vanish in general.

We are interested in the reduction to $A^{(1)}_1$, which is the invariant subalgebra of $gl(\infty)$ by the automorphism of the free fermion algebra,

$$\sigma\begin{cases}\psi^{(\alpha)}(k)\mapsto k\psi^{(\alpha)}(k)\\ \psi^{*(\alpha)}(k)\mapsto k^{-1}\psi^{*(\alpha)}(k).\end{cases} \tag{6}$$

This means that $a^{(\alpha,\beta)}_{i,j} = a^{(\alpha,\beta)}_{i-j}$. Thus any element of the corresponding subgroup, g, satisfies

$$\sum_{\alpha=1,2}\sum_{i\in\mathbb{Z}+1/2}\psi^{(\alpha)}_{-i}g\otimes\psi^{*(\alpha)}_{i+s}g = \sum_{\alpha=1,2}\sum_{i\in\mathbb{Z}+1/2} g\psi^{(\alpha)}_{-i}\otimes g\psi^{*(\alpha)}_{i+s} \tag{7}$$

for any integer s. Furthermore, because g commutes with $\mathcal{H}(x,x)$ and $\bar{\mathcal{H}}(\bar{x},\bar{x})$, the vacuum expectation value thus enjoys the following properties:

$$\begin{aligned}&\langle s_1,s_2|g(x^{(1)},x^{(2)},\bar{x}^{(1)},\bar{x}^{(2)})|t_2,t_1\rangle\\ &\qquad = (-)^{(s_1+t_1)l}\langle s_1-l,s_2-l|g(x^{(1)},x^{(2)},\bar{x}^{(1)},\bar{x}^{(2)})|t_2-l,t_1-l\rangle\end{aligned} \tag{8a}$$

$$\begin{aligned}&\langle s_1,s_2|g(x^{(1)},x^{(2)},\bar{x}^{(1)},\bar{x}^{(2)})|t_2,t_1\rangle = \langle s_1,s_2|g(x^{(1)}-x,x^{(2)}-x,\bar{x}^{(1)}-\bar{x},\bar{x}^{(2)}-\bar{x})|t_2,t_1\rangle\\ &\qquad\times\exp\left(2\eta(x,\bar{x})-\sum_{\alpha=1,2}[\eta(x^{(\alpha)},\bar{x})+\eta(x,\bar{x}^{(\alpha)})]\right)\end{aligned} \tag{8b}$$

where

$$g(x^{(1)},x^{(2)},\bar{x}^{(1)},\bar{x}^{(2)}) = \mathrm{e}^{\mathcal{H}(x^{(1)},x^{(2)})}g\,\mathrm{e}^{-\bar{\mathcal{H}}(\bar{x}^{(1)},\bar{x}^{(2)})} \qquad \eta(x,\bar{x}) = \sum_{m=1}^{\infty}mx_m\bar{x}_m.$$

Consequently, a nontrivial expectation value is characterized by two integers a and b

$$a = s_1 - t_1 \qquad b = s_2 - t_1$$

and a pair of series of times, z and $\bar{z}$

$$z = x^{(1)} - x^{(2)} = (z_1, z_2, \ldots) \qquad \bar{z} = \bar{x}^{(1)} - \bar{x}^{(2)} = (\bar{z}_1, \bar{z}_2, \ldots).$$

For convenience, we call z ($\bar{z}$) the positive (negative) series of times. The τ function $\tau_{a,b}(z, \bar{z})$ introduced by

$$\tau_{a,b}(z, \bar{z}) = \langle a, b | g(z, 0, \bar{z}, 0) | a + b, 0\rangle \tag{9}$$

distinguishes the expectation value up to properties (8).

For example, in the case where $a = 0, \pm 1$ and

$$g = \exp\left(\sum_{i=1}^{N} \alpha_i \psi^{(1)}(p_i)\psi^{*(2)}(p_i) + \beta_i \psi^{(2)}(q_i)\psi^{*(1)}(q_i)\right) \tag{10}$$

the τ functions are calculated as

$$\tau_{0,n} \exp(\eta(z, \bar{z})) = \sum_{r=0}^{N} \sum_{1\leqslant i_1 < i_2 \ldots i_r \leqslant N} \sum_{1\leqslant j_1 < j_2 \ldots j_r \leqslant N} \times \prod_{\mu<\nu\leqslant r} c(i_\mu, i_\nu, j_\mu)\, \bar{c}(i_\mu, j_\mu, j_\nu) \prod_{\nu=1}^{r} d(i_\nu, j_\nu) E_{n,i_\nu} \bar{E}_{n,j_\nu} \tag{11a}$$

$$\tau_{-1,n} \exp(\eta(z, \bar{z})) = (-)^{n-1} \sum_{r=1}^{N} \sum_{1\leqslant i_1 < i_2 \ldots i_{r-1} \leqslant N} \sum_{1\leqslant j_1 < j_2 \ldots j_r \leqslant N} \times \prod_{\mu<\nu\leqslant r-1} c(i_\mu, i_\nu, j_\mu) \prod_{\mu<\nu\leqslant r} \bar{c}(i_\mu, j_\mu, j_\nu) \prod_{\nu=1}^{r-1} d(i_\nu, j_\nu) E_{n,i_\nu} \prod_{\nu=1}^{r} \bar{E}_{n,j_\nu} \tag{11b}$$

$$\tau_{1,n} \exp(\eta(z, \bar{z})) = (-)^{n} \sum_{r=1}^{N} \sum_{1\leqslant i_1 < i_2 \ldots i_r \leqslant N} \sum_{1\leqslant j_1 < j_2 \ldots j_{r-1} \leqslant N} \times \prod_{\mu<\nu\leqslant r} c(i_\mu, i_\nu, j_\mu) \prod_{\mu<\nu\leqslant r-1} \bar{c}(i_\mu, j_\mu, j_\nu) \prod_{\nu=1}^{r} E_{n,i_\nu} \prod_{\nu=1}^{r-1} d(i_\nu, j_\nu) \bar{E}_{n,j_\nu} \tag{11c}$$

where

$$c(i_\mu, i_\nu, j_\mu) = \frac{(p_{i_\mu} - p_{i_\nu})^2}{(q_{j_\mu} - p_{i_\nu})^2} \qquad \bar{c}(i_\mu, j_\mu, j_\nu) = \frac{(q_{j_\mu} - q_{j_\nu})^2}{(p_{i_\mu} - q_{j_\nu})^2}$$

$$d(i_\mu, j_\mu) = \frac{1}{(p_{i_\mu} - q_{j_\mu})^2}$$

$$E_{n,i} = p_i^{-n} \alpha_i \,\mathrm{e}^{\xi(z,p_i)+\xi(\bar{z},p_i^{-1})} \qquad \bar{E}_{n,j} = q_j^n \beta_j \,\mathrm{e}^{-\xi(z,q_j)-\xi(\bar{z},q_j^{-1})}.$$

Here we used the formula

$$\langle \mathrm{vac}| \,\mathrm{e}^{\mathcal{H}(z,0)}\, \mathrm{e}^{-\bar{\mathcal{H}}(\bar{z},0)} |\mathrm{vac}\rangle = \exp(-\eta(z, \bar{z})). \tag{12}$$

According to the above observations, the bilinear identity for the $A_1^{(1)}$ reduction of the 2cTL hierarchy is as follows: for any bras $\langle u|$ and $\langle u'|$, any kets $|v\rangle$ and $|v'\rangle$, and any integer s,

$$\sum_{\alpha=1,2}\oint\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\lambda^s\langle u|\psi^{(\alpha)}(\lambda)g|v\rangle\langle u'|\psi^{*(\alpha)}(\lambda)g|v'\rangle$$

$$=\sum_{\alpha=1,2}\oint\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\lambda^s\langle u|g\psi^{(\alpha)}(\lambda)|v\rangle\langle u'|g\psi^{*(\alpha)}(\lambda)|v'\rangle \tag{13}$$

where the contour of the integration of the lhs is a small circle around $\lambda=\infty$, and that of the rhs is a small circle around $\lambda=0$.

3. Derivation of AL hierarchy from 2cTL hierarchy

In this section, we shall show that all the equations of the AL hierarchy are derived from (13), i.e., the $A_1^{(1)}$ reduction of the 2cTL hierarchy includes the AL hierarchy.

We illustrate the derivation of the functional representation and the Lax representation of the AL hierarchy associated with the positive series of times, z, from the $A_1^{(1)}$ reduction of the 2cTL hierarchy. Let s_α, t_α, s'_α and t'_α $(\alpha=1,2)$ be integers such that

$$s_1+s_2=t_1+t_2-1 \qquad s'_1+s'_2=t'_1+t'_2+1.$$

For

$$\langle u|=\langle s_1+2,s_2+1|\,\mathrm{e}^{\mathcal{H}(x^{(1)}+x,x^{(2)}+y)}\psi^{(1)}(k) \qquad |v\rangle=\mathrm{e}^{-\bar{\mathcal{H}}((\bar{x}^{(1)}+\bar{x},\bar{x}^{(2)}+\bar{y}))}|t_2,t_1\rangle$$

$$\langle u'|=\langle s'_1-1,s'_2-1|\,\mathrm{e}^{\mathcal{H}(x^{(1)}-x,x^{(2)}-y)} \qquad |v'\rangle=\mathrm{e}^{-\bar{\mathcal{H}}(\bar{x}^{(1)}-\bar{x},\bar{x}^{(2)}-\bar{y})}|t'_2,t'_1\rangle$$

equation (13) demands the bilinear identity

$$\begin{aligned}
&(-)^{b+b'+1}k^{-1}\sum_{l=0}^{\infty}p_l(2x)p_{l+s+s_1-s'_1+2}(2\bar{y}-\tilde{D})\,\mathrm{e}^{d(x,y,\bar{x},\bar{y})}\tau_{a,b+1}(z-\epsilon(k^{-1}))\tau_{a',b'-1}\\
&\quad+(-)^{b+b'}\sum_{l=0}^{\infty}p_l(2x)p_{l+s+s_1-s'_1+1}(2\bar{y}-\tilde{D})\,\mathrm{e}^{d(x,y,\bar{x},\bar{y})}\tau_{a,b+1}(z-\epsilon(k^{-1}))\tau_{a',b'-1}\\
&\quad+\sum_{l=0}^{\infty}p_l(2y)p_{l+s+s_2-s'_2+1}(2\bar{x}+\tilde{D})\,\mathrm{e}^{d(x,y,\bar{x},\bar{y})}\tau_{a+1,b}(z-\epsilon(k^{-1}))\tau_{a'-1,b'}\\
&=(-)^{b+b'}\sum_{l=0}^{\infty}p_l(2\bar{x})p_{l-(s+t_1-t'_1+1)}(2y-\tilde{\bar{D}})\,\mathrm{e}^{d(x,y,\bar{x},\bar{y})}\tau_{a,b}(z-\epsilon(k^{-1}))\tau_{a',b'}\\
&\quad+\sum_{l=0}^{\infty}p_l(2\bar{y})p_{l-(s+t_2-t'_2+1)}(2x+\tilde{\bar{D}})\,\mathrm{e}^{d(x,y,\bar{x},\bar{y})}\tau_{a+1,b+1}(z-\epsilon(k^{-1}))\tau_{a'-1,b'-1}\\
&\quad-k^{-1}\sum_{l=0}^{\infty}p_l(2\bar{y})p_{l-(s+t_2-t'_2+2)}(2x+\tilde{\bar{D}})\,\mathrm{e}^{d(x,y,\bar{x},\bar{y})}\tau_{a+1,b+1}(z-\epsilon(k^{-1}))\tau_{a'-1,b'-1}
\end{aligned} \tag{14}$$

where $\tilde{D}=\left(D_1,\frac{D_2}{2},\ldots\right)$, $\tilde{\bar{D}}=\left(\bar{D}_1,\frac{\bar{D}_2}{2},\ldots\right)$ (D_m and $\bar{D}_m$ refer to the Hirota operators with respect to z_m and $\bar{z}_m$, respectively), $\epsilon(k)=(k,k^2/2,\ldots,k^m/m,\ldots)$,

$$d(x,y,\bar{x},\bar{y})=\sum_{m=1}^{\infty}(x_m-y_m)D_m+(\bar{x}_m-\bar{y}_m)\bar{D}_m \tag{15}$$

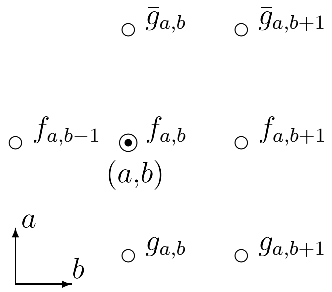

Figure 1. The locations of the τ functions.

and $p_l(x)$ are the bases of the Schur polynomials that are defined by

$$e^{\xi(x,k)} = \sum_{l=0}^{\infty} p_l(x) k^l \tag{16}$$

with $\xi(x,k) = \sum_{n=1}^{\infty} x_n k^n$.

We focus our attention on the following combination of the τ functions for integers a and n:

$$f_{a,n} = \tau_{a,n} \qquad g_{a,n} = (-)^n \tau_{a-1,n} \qquad \bar{g}_{a,n} = (-)^n \tau_{a+1,n} \tag{17}$$

(see figure 1). Substituting $k = \zeta^{-2}$ in (14) we infer that

$$f_{a,n}^{-\epsilon} f_{a,n} = g_{a,n}^{-\epsilon} \bar{g}_{a,n} + f_{a,n+1}^{-\epsilon} f_{a,n-1} \tag{18a}$$

$$\zeta^2 g_{a,n+1}^{-\epsilon} f_{a,n-1} = g_{a,n}^{-\epsilon} f_{a,n} - f_{a,n}^{-\epsilon} g_{a,n} \tag{18b}$$

$$-\zeta^2 f_{a,n+1} \bar{g}_{a,n}^{+\epsilon} = -f_{a,n} \bar{g}_{a,n+1}^{+\epsilon} + \bar{g}_{a,n+1} f_{a,n}^{+\epsilon} \tag{18c}$$

$$-\zeta^2 g_{a,n+1} \bar{g}_{a,n}^{+\epsilon} = -f_{a,n} f_{a,n+1}^{+\epsilon} + f_{a,n+1} f_{a,n}^{+\epsilon} \tag{18d}$$

where $f_{a,n}^{\pm\epsilon} = f_{a,n}(z \pm \epsilon(\zeta^2), \bar{z})$. The quadratic equations (18) are the key equations of this paper because they can be written in the form of the scattering problem of the AL hierarchy as follows. Let us introduce the dependent variables of the AL hierarchy, $\rho_{a,n}$ and $\sigma_{a,n}$, as

$$\rho_{a,n} = \frac{g_{a,n+1}}{f_{a,n+1}} \qquad \sigma_{a,n} = \frac{\bar{g}_{a,n+1}}{f_{a,n+1}} \tag{19}$$

and consider the 2×2 matrices

$$\Psi_{a,n}^{(+)}(\zeta) = \frac{1}{f_{a,n}} \begin{pmatrix} f_{a,n+1}(z - \epsilon(\zeta^2), \bar{z}) & \zeta \bar{g}_{a,n}(z + \epsilon(\zeta^2), \bar{z}) \\ -\zeta g_{a,n+1}(z - \epsilon(\zeta^2), \bar{z}) & f_{a,n}(z + \epsilon(\zeta^2), \bar{z}) \end{pmatrix} \Psi_{\mathrm{vac},n}^{(+)}(\zeta) \tag{20}$$

$$\Psi_{\mathrm{vac},n}^{(+)}(\zeta) = \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix} \begin{pmatrix} e^{\xi(z,\zeta^{-2})} & 0 \\ 0 & e^{\xi(\bar{z},\zeta^2)} \end{pmatrix} \tag{21}$$

$$L_{a,n}(\zeta) = \begin{pmatrix} \zeta & \sigma_{a,n} \\ \rho_{a,n} & \zeta^{-1} \end{pmatrix}. \tag{22}$$

Then equations (18) are read as

$$\Psi_{n+1}^{(+)} = L_n \Psi_n^{(+)} \tag{23}$$

which is nothing but the scattering problem of the AL hierarchy [5], and $\Psi_n^{(+)}$ is called a wavefunction of the problem.

In addition, the functional representation [11] of the AL hierarchy is derived from (18) in the following way. Let $d_{a,n}$ and $\bar{d}_{a,n}$ stand for the additional dependent variables defined by

$$d_{a,n} = \frac{f_{a,n+1}}{f_{a,n}} \qquad \bar{d}_{a,n} = \frac{1}{d_{a,n}}. \tag{24}$$

With $\rho_{a,n}$, $\sigma_{a,n}$, $d_{a,n}$ and $\bar{d}_{a,n}$, equations (18*a*) and (18*b*) are then written as

$$\rho_{a,n}\sigma_{a,n}^{+\epsilon} + d_{a,n+1}\bar{d}_{a,n}^{+\epsilon} = 1 \tag{25a}$$

$$\rho_{a,n}^{+\epsilon} - \rho_{a,n} + \zeta^2\rho_{a,n+1}d_{a,n+1}\bar{d}_{a,n}^{+\epsilon} = 0 \tag{25b}$$

$$\sigma_{a,n}^{+\epsilon} - \sigma_{a,n} - \zeta^2\sigma_{a,n-1}^{+\epsilon}d_{a,n+1}\bar{d}_{a,n}^{+\epsilon} = 0. \tag{25c}$$

Because (18*c*) and (18*d*) can be derived from other two, equations (25) are equivalent to equations (18). Below, we drop the index a from the subscript of the notation when there is no chance of confusion. Using (25*a*) one can eliminate d and $\bar{d}$ from (25*b*) and (25*c*). One thus obtains the functional representation of (the 'positive part' of) the AL hierarchy, which is nothing but the spacetime discretized NLS equation by Suris [6],

$$\rho_n^{+\epsilon} - \rho_n + \zeta^2\rho_{n+1}\big(1 - \rho_n\sigma_n^{+\epsilon}\big) = 0 \tag{26a}$$

$$\sigma_n^{-\epsilon} - \sigma_n - \zeta^2\sigma_{n-1}\big(1 - \rho_n^{-\epsilon}\sigma_n\big) = 0. \tag{26b}$$

The expansion of equations (26) in powers of ζ^2 yields the hierarchy of equations.

The hierarchy and other relations for the negative series of times, $\bar{z}$, can be obtained in the same manner as the positive one by substituting in (13)

$$\langle u| = \langle s_1+1, s_2+1|\,\mathrm{e}^{\mathcal{H}(x^{(1)}+x,x^{(2)}+y)} \qquad |v\rangle = \psi^{(1)}(k)\,\mathrm{e}^{-\bar{\mathcal{H}}((\bar{x}^{(1)}+\bar{x},\bar{x}^{(2)}+\bar{y}))}|t_2, t_1-1\rangle$$

$$\langle u'| = \langle s_1'-1, s_2'-1|\,\mathrm{e}^{\mathcal{H}(x^{(1)}-x,x^{(2)}-y)} \qquad |v'\rangle = \mathrm{e}^{-\bar{\mathcal{H}}(\bar{x}^{(1)}-\bar{x},\bar{x}^{(2)}-\bar{y})}|t_2', t_1'\rangle.$$

We can propose two Bäcklund transformations of the AL hierarchy. The first one can be deduced from the introduction of the so-called Hirota–Miwa variables [15]. This is often called the discretization of the time, and is already obtained as (26). Another corresponds to the change of the index a. The index a can take any integer value in our theory: if $\tau_{a,n}$ is the τ function of the AL hierarchy, so is $\tau_{a+1,n}$. To formulate this transformation, one uses equation (18*a*) with $\zeta = 0$ and obtains

$$g_{a+1,n} = f_{a,n} \tag{27a}$$

$$f_{a+1,n} = \bar{g}_{a,n} \tag{27b}$$

$$\bar{g}_{a+1,n} = \frac{1}{f_{a,n}}\left(\bar{g}_{a,n}^2 - \bar{g}_{a,n+1}\bar{g}_{a,n-1}\right). \tag{27c}$$

Although (18*a*) are known to have a relation to the discrete Toda equations and its variants, the author believes that it is the first time this equation has appeared in this context.

The evolution equations of $\Psi_n^{(\pm)}$ with respect to z_m and $\bar{z}_m$ ($m = 1, 2, \ldots$) are also obtained from the bilinear identity (13):

$$\left(\frac{\partial}{\partial z_m} - V_n^{(+m)}\right)\Psi_n^{(\pm)} = 0 \qquad \left(\frac{\partial}{\partial \bar{z}_m} - V_n^{(-m)}\right)\Psi_n^{(\pm)} = 0. \tag{28}$$

For instance, the first two are

$$V_n^{(+1)}(\zeta) = \begin{pmatrix} \zeta^{-2} & -\zeta^{-1}\sigma_{n-1} \\ -\zeta^{-1}\rho_n & \rho_n\sigma_{n-1} \end{pmatrix} \tag{29a}$$

$$V_n^{(-1)}(\zeta) = \begin{pmatrix} \zeta^2 - \rho_{n-1}\sigma_n & \zeta\sigma_n \\ \zeta\rho_{n-1} & 0 \end{pmatrix} \tag{29b}$$

$$V_n^{(2)}(\zeta) = \zeta^{-2} V_n^{(1)}(\zeta) + \begin{pmatrix} -\zeta^{-2}\rho_n\sigma_{n-1} & -\zeta^{-1}\left(\sigma_{n-2}\pi_{n-1} - \rho_n\sigma_{n-1}^2\right) \\ \zeta^{-1}\left(\rho_n^2\sigma_{n-1} - \rho_{n+1}\pi_n\right) & \rho_{n+1}\sigma_{n-1}\pi_n - \rho_n^2\sigma_{n-1}^2 + \rho_n\sigma_{n-2}\pi_{n-1} \end{pmatrix} \tag{29c}$$

$$V_n^{(-2)}(\zeta) = \zeta^2 V_n^{(-1)}(\zeta) + \begin{pmatrix} -\rho_{n-1}\sigma_{n+1}\pi_n + \rho_{n-1}^2\sigma_n^2 - \rho_{n-2}\sigma_n\pi_{n-1} & \zeta\left(\sigma_{n+1}\pi_n - \rho_{n-1}\sigma_n^2\right) \\ \zeta\left(\rho_{n-2}\pi_{n-1} - \rho_{n-1}^2\sigma_n\right) & \zeta^2\rho_{n-1}\sigma_n \end{pmatrix} \tag{29d}$$

where $\pi_n = 1 - \rho_n\sigma_n$. The compatibility conditions of these equations are ensured by the bilinear identity.

We summarize the result of this section. For the $A_1^{(1)}$ reduction of the 2cTL hierarchy, three τ functions f, g and $\bar{g}$ are defined by the vacuum expectation values of free fermions. The bilinear identity for them (14) then includes equations (18), which are equivalent to the scattering problem and the functional representation of the AL hierarchy. The spectral parameter of the 2cTL hierarchy, k relates to that of the AL hierarchy, ζ, by the relation $k = \zeta^{-2}$.

Before concluding this section, we give some comments on the τ function of other versions of the NLS equations. Date *et al* derived the (continuous) NLS equation and the full discrete NLS equation [15] from the $A_1^{(1)}$ reduction of the two-component KP hierarchy. On the other hand, we discussed the sdNLS equation and its τ functions. In fact, our τ functions for $\bar{z} = (0, 0, \ldots)$ are identical with those of the $A_1^{(1)}$ reduction of the two-component KP hierarchy. According to Date *et al*, $\bar{g}_{n-1}/f_n$ and g_{n+1}/f_n satisfy the continuous NLS equation and the full discrete NLS equation, while $\bar{g}_n/f_n$ and g_n/f_n solve the sdNLS equation.

4. Relation to RT hierarchy

It is known from the discussion in the framework of the one-component Toda lattice hierarchy that the AL hierarchy and the RT hierarchy are equivalent [13]. In this section, we briefly review the equivalence of the AL hierarchy and the RT hierarchy in our framework, so that we can propose the τ function of the RT hierarchy as the vacuum expectation value of free fermions.

Let us introduce a new matrix function $\Phi_n^{(\pm)}$ using the wavefunction of the AL hierarchy, $\Psi_n^{(\pm)}$ as

$$\Phi_n^{(\pm)}(\zeta) = \mathrm{diag}(\zeta^n, \zeta^{-n})\Psi_n^{(\pm)} \exp(-\xi(\bar{z}, \zeta^2)). \tag{30}$$

Then, from (23) and the 'negative' one, we acquire the three-term recurrent relation

$$\Phi_{n+1}^{(\pm)} + A_n\Phi_n^{(\pm)} = \mathrm{diag}(\zeta^2, \zeta^{-2})\left(\Phi_n^{(\pm)} + B_n\Phi_{n-1}^{(\pm)}\right). \tag{31}$$

Here, $A_n = \mathrm{diag}\left(-\frac{\sigma_n}{\sigma_{n-1}}, -\frac{\rho_n}{\rho_{n-1}}\right)$ and $B_n = \pi_{n-1}A_n$. Equations (28) yield the time evolutions of $\Phi_n^{(\pm)}$ with respect to z_1 and $\bar{z}_1$,

$$\partial_1\Phi_n^{(\pm)} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\rho_n}{\rho_{n-1}}\pi_{n-1} \end{pmatrix}\Phi_n^{(\pm)} + \pi_{n-1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho_n}{\rho_{n-1}}\zeta^{-2} \end{pmatrix}\Phi_{n-1}^{(\pm)} \tag{32a}$$

$$\bar{\partial}_1\Phi_n^{(\pm)} = \begin{pmatrix} \frac{\sigma_n}{\sigma_{n-1}}\pi_{n-1} & 0 \\ 0 & 0 \end{pmatrix}\Phi_n^{(\pm)} + \pi_{n-1}\begin{pmatrix} -\frac{\sigma_n}{\sigma_{n-1}}\zeta^2 & 0 \\ 0 & -1 \end{pmatrix}\Phi_{n-1}^{(\pm)}. \tag{32b}$$

Note that all the coefficient matrices are diagonal so that the relations are 'scalar-like'.

We shall show that each of the compatibility conditions for equations (31) and (32*a*), and that of equations (31) and (32*b*) is equivalent to the RT equation. From now on, we concentrate on the equations for the first row of $\Phi_n^{(\pm)}$. Let w_n stand for an element in the first row of $\Phi_n^{(\pm)}$, $a_n = -\frac{\sigma_n}{\sigma_{n-1}}$, and $b_n = a_n\pi_{n-1}$. The first rows of equations (31) and (32*a*) are then written as

$$w_{n+1} + a_n w_n = \zeta^2(w_n + b_n w_{n-1}) \qquad \partial w_n = \frac{b_n}{a_n} w_{n-1}. \tag{33}$$

Replacing the dependent variables as

$$a_n = \exp(-\epsilon\theta_n) \qquad b_n = -c^{-2}\exp(\phi_n - \phi_{n-1} - \epsilon\theta_n) \tag{34}$$

we obtain the compatibility condition for (33):

$$\partial_1\phi_n = -[1 + \epsilon^2\exp(\phi_{n+1} - \phi_n)]\exp(\epsilon\theta_n) + \nu_1 \tag{35a}$$

$$\partial_1\theta_n = \frac{-1}{\epsilon c^2}[\exp(\phi_{n+1} - \phi_n + \epsilon\theta_n) - \exp(\phi_n - \phi_{n-1} + \epsilon\theta_{n-1})]. \tag{35b}$$

Here ν_1 is a constant that is determined by the boundary condition. Eliminating θ_n from (35), we acquire the RT equation

$$\begin{aligned}\partial_1^2\phi_n = (\nu_1 - \partial_1\phi_{n+1})(\nu_1 - \partial_1\phi_n)&\frac{c^{-2}\exp(\phi_{n+1} - \phi_n)}{1 + c^{-2}\exp(\phi_{n+1} - \phi_n)}\\ - (\nu_1 - \partial_1\phi_n)(\nu_1 - \partial_1\phi_{n-1})&\frac{c^{-2}\exp(\phi_n - \phi_{n-1})}{1 + c^{-2}\exp(\phi_n - \phi_{n-1})}\end{aligned} \tag{36}$$

where c denotes the light speed. With $t = \nu_1 z_1/c$,

$$\begin{aligned}\partial_t^2\phi_n = (1 - c^{-1}\partial_t\phi_{n+1})(1 - c^{-1}\partial_t\phi_n)&\frac{\exp(\phi_{n+1} - \phi_n)}{1 + c^{-2}\exp(\phi_{n+1} - \phi_n)}\\ - (1 - c^{-1}\partial_t\phi_n)(1 - c^{-1}\partial_t\phi_{n-1})&\frac{\exp(\phi_n - \phi_{n-1})}{1 + c^{-2}\exp(\phi_n - \phi_{n-1})}.\end{aligned} \tag{37}$$

On the other hand, the first rows in equations (31) and (32*b*) can be written as

$$w_{n+1} + a_n w_n = \zeta^2(w_n + b_n w_{n-1}) \qquad \bar{\partial}_1 w_n = -b_n(w_n - \zeta^{-2}w_{n-1}) \tag{38}$$

where w_n is an element in the first row of $\Phi_n^{(\pm)}$. After the exchange of the variables by (34), the compatibility condition for (38) is written as

$$\bar{\partial}_1\phi_n = [1 + \epsilon^2\exp(\phi_n - \phi_{n-1})]\exp(-\epsilon\theta_n) + \nu_2 \tag{39a}$$

$$\bar{\partial}_1\theta_n = -c^{-2}[\exp(\phi_{n+1} - \phi_n - \epsilon\theta_{n+1}) - \exp(\phi_n - \phi_{n-1} - \epsilon\theta_n)]. \tag{39b}$$

Eliminating θ_n from (39) and replacing the independent variable as $t = \nu_2\bar{z}_1/c$, we again obtain (37).

The compatibility condition for the element in the second row is shown to be the RT equation in the same manner.

As a result, the $\Phi_n^{(\pm)}$ is the wavefunction of the RT hierarchy, and consequently the AL hierarchy derives the RT hierarchy.

Next, we show that the RT hierarchy implies the AL hierarchy. Let Φ_n be given as a wavefunction of the RT hierarchy. Consider

$$\tilde{\Psi}_n = Z^{-n}\Phi_n \qquad \tilde{\Psi}'_{n+1} = \sum_{l=-\infty}^{n} Z^{n-l}Q_{l-1}\tilde{\Psi}_{l-1}$$

$$Z = \mathrm{diag}(\zeta, \zeta^{-1}) \qquad Q_n = \begin{pmatrix} 0 & \tilde{\sigma}_n \\ \tilde{\rho}_n & 0 \end{pmatrix}$$

$$\tilde{\sigma}_n = (-)^n \prod_{n'=-\infty}^{n} (A_{n'})_{11} \qquad \tilde{\rho}_n = (-)^n \prod_{n'=-\infty}^{n} (A_{n'})_{22}\,.$$

Then the matrices $\tilde{\Psi}_n$ and $\tilde{\Psi}'_n$ satisfy that

$$\tilde{\Psi}_{n+1} = Z\tilde{\Psi}_n + Q_n\tilde{\Psi}'_n \qquad \tilde{\Psi}'_{n+1} = Z\tilde{\Psi}'_n + Q_n\tilde{\Psi}_n. \tag{40}$$

If $\tilde{\Psi}_n = \tilde{\Psi}'_n$, then each of equations (44) is nothing but the scattering problem of the AL hierarchy. Also, C_n defined by

$$C_n = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} (\tilde{\Psi}_n - \tilde{\Psi}'_n) \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{41}$$

satisfies it. Therefore the RT hierarchy is equivalent to the AL hierarchy.

On the basis of these observations, one can establish the relation between the solution of the RT equation, ϕ_n, and the τ function. Equations (34) imply that

$$\frac{b_n}{a_n} = -c^{-2} \exp(\phi_n - \phi_{n-1}) \tag{42}$$

whereas the definition yields the identity

$$\frac{b_n}{a_n} = \pi_{n-1} = \frac{f_{n+1} f_{n-1}}{f_n^2}. \tag{43}$$

Therefore with $\tau_{a,n}$ given by (9),

$$\exp \phi_n = (-c^2)^n \frac{\tau_{a,n+1}}{\tau_{a,n}}. \tag{44}$$

5. Conclusion and discussion

We have shown in the language of free fermions that the $A_1^{(1)}$ reduction of the 2cTL hierarchy includes the AL (RT) hierarchy. In other words, all the equations that belong to the AL hierarchy (the RT hierarchy) are deduced from the $A_1^{(1)}$ reduction of the 2cTL hierarchy. We focused our attention on three τ functions, and showed that they solve the equations of the AL hierarchy and the RT hierarchy. A Bäcklund transformation for the AL hierarchy is also presented.

Our result implies that the 'universality' [10] of the AL hierarchy stems from the fact that the 2cTL hierarchy is not only a multicomponent generalization of the Toda lattice hierarchy, but also a generalization of the two-component KP hierarchy with 'negative series of time'.

The Casoratian solution of the RT equation is constructed by Ohta *et al* [16]. In general, the bilinear equation expression of a nonlinear equation is not unique. Because the bilinear equation of the RT equation in our paper is different from theirs, the correspondence between their τ functions and ours is not simple. This is an open problem.

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